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GENERAL THEORY OF ASYMMETRIC WAVES IN A CIRCULAR WAVEGUIDE WITH AN OPEN END

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GENERAL THEORY OF ASYMMETRIC WAVES IN A CIRCULAR WAVEGUIDE WITH AN OPEN END

L. A. Vaynshteyn

A rigorous solution is obtained for the problem in which an asymmetric electric or magnetic wave propagates inside a circular waveguide towards its open end (semi-infinite tube). Since the electromagnetic field produced by the diffraction of this wave at the end of the tube has two Hertz functions, the solution is considerably more complicated than for the symmetric waves. Equations are presented for the reflection coefficient of the incident wave from the open end, and also equations for the transformation coefficients of this wave into other waves (including the transformation coefficients of electric waves into magnetic waves and vice versa). The radiation field is investigated and design equations are presented for the radiation characteristics. Approximate equations are derived for the radiation field and for the reflection and transformation coefficients.

Section 1. Formulation and General Solution of the Problem

The problem of asymmetric waves in a circular waveguide with an open end /328* is formulated in the same way as the problem for symmetric waves (ref. 1). We consider a semi-infinite cylindrical tube whose lateral surface is situated at $r=a$, $z>0$ (in a cylindrical system of coordinates r, φ, z). An electric wave E_{pl} or a magnetic wave H_{pl} propagates inside the tube towards the open end which is at $z=0$. Our purpose is to compute the electromagnetic field which results from the diffraction of this wave at the open end of the waveguide. Nonsymmetric electric and magnetic waves ($p=1, 2, 3, \ldots$) are distinguished from symmetric waves ($p=0$) which we considered earlier, by the fact that the diffraction field of nonsymmetric waves is characterized by two scalar functions ("the Hertz functions"). The necessity of introducing these functions will be clear from our presentation below. At this time we shall assume that the longitudinal component of the electric Hertz vector is equal to

$$\Pi_{ez} = \sin(p\varphi + \varphi_0)\,\Pi(r, z), \tag{1}$$

while the longitudinal component of the magnetic Hertz vector is equal to

$$\Pi_{mz} = \cos(p\varphi + \varphi_0)\,\tilde{\Pi}(r, z). \tag{2}$$

The electromagnetic fields are expressed in terms of the functions Π and $\tilde{\Pi}$ as follows (compare ref. 2, Chapter 8):

*Numbers given in margin indicate pagination in original foreign text.

$$\left.\begin{array}{ll} E_r=\sin(p\varphi+\varphi_0)\left[\frac{\partial^2\Pi}{\partial r\,\partial z}-\frac{ipk}{r}\tilde{\Pi}\right] & H_r=\cos(p\varphi+\varphi_0)\left[-\frac{ipk}{r}\Pi+\frac{\partial^2\tilde{\Pi}}{\partial r\,\partial z}\right] \\ E_\varphi=\cos(p\varphi+\varphi_0)\left[\frac{p}{r}\frac{\partial\Pi}{\partial z}-ik\frac{\partial\tilde{\Pi}}{\partial r}\right] & H_\varphi=\sin(p\varphi+\varphi_0)\left[ik\frac{\partial\Pi}{\partial r}-\frac{p}{r}\frac{\partial\tilde{\Pi}}{\partial z}\right] \\ E_z=\sin(p\varphi+\varphi_0)\left(\frac{\partial^2}{\partial z^2}+k^2\right)\Pi & H_z=\cos(p\varphi+\varphi_0)\left(\frac{\partial^2}{\partial z^2}+k^2\right)\tilde{\Pi} \end{array}\right\} \quad (3)$$

The constant angle φ_0 is determined by the polarization of the wave incident at the open end. The time function is taken in the form $e^{-i\omega t}=e^{-ickt}$. /329

Functions Π and $\tilde{\Pi}$ must be solutions of the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Pi}{\partial r}\right)+\frac{\partial^2\Pi}{\partial z^2}+\left(k^2-\frac{p^2}{r^2}\right)\Pi=0,$$

therefore, we seek them in the form

$$\left.\begin{array}{l} \Pi(r,\ z)=-\frac{2\pi^2 a}{ck}\int\limits_C e^{iwz}\begin{Bmatrix}J_p(vr)\,H_p(va)\\ J_p(va)\,H_p(vr)\end{Bmatrix}F(w)\,dw \\ \tilde{\Pi}(r,\ z)=-\frac{i2\pi^2 a}{c}\int\limits_C e^{iwz}\begin{Bmatrix}J_p(vr)\,H_p'(va)\\ J_p'(va)\,H_p(vr)\end{Bmatrix}\frac{1}{v}G(w)\,dw \end{array}\right\}. \quad (4)$$

Here C is a contour in the plane of the complex variable w which (compare ref. 1, p. 1546) we trace primarily along the real axis and below the point (see below) which corresponds to the wave number of the incident waves; J_p and $H_p=H_p^{(1)}$ are Bessel and Hankel functions, $v=\sqrt{k^2-w^2}$, $\operatorname{Im} v>0$ (for $\operatorname{Im} k>0$).

The upper line of (4) should be taken when r<a, and the lower line should be taken when r>a.

The vector of the surface current density on the walls of the tube has the components

$$\left.\begin{array}{l} j_\varphi=\cos(p\varphi+\varphi_0)\int\limits_C e^{iwz}G(w)\,dw \\ j_z=\sin(p\varphi+\varphi_0)\int\limits_C e^{iwz}\left[F(w)+\frac{ip}{a}\frac{w}{v^2}G(w)\right]dw \end{array}\right\}. \quad (5)$$

The functions F(w) and G(w), which remain unknown, should be determined from three considerations. The first of these is that the tangential components H_z and H_φ of the magnetic field must be continuous when $r=\alpha$, $z<0$, i.e., on the extension of the tube's wall we must have

$$j_\varphi=j_z=0.$$

This gives us relationships

$$\int_C e^{iwz} G(w)\,dw = 0 \quad \text{for} \quad z < 0, \tag{6}$$

$$\int_C e^{iwz}\left[F(w) + \frac{ip}{a}\frac{w}{v^2}G(w)\right]dw = 0 \quad \text{for} \quad z \leqslant 0, \tag{7}$$

where the latter must also be valid when z=0 because the component j_z of the surface current density at the edge of the wall (when z=+0 (sic)) must vanish. We note that the continuity of the tangential components E_z and E_φ of the electric field when r=a are provided by the very expressions (4) for Π and $\tilde{\Pi}$ written in such a way so as to take this requirement into consideration.

In the second place the following boundary conditions must be satisfied on the wall of the waveguide which we consider to be an ideal conductor: /330

$$E_z = E_\varphi = 0 \quad \text{for} \quad r = a,\ z > 0,$$

and these lead us to relationships

$$\int_C e^{iwz} v\varphi(wa) F(w)\,dw = 0 \quad \text{for} \quad z > 0, \tag{8}$$

$$\int_C e^{iwz}\left[\frac{\psi(wa)}{v}G(w) + \frac{ip}{k^2 a}\frac{w\varphi(wa)}{v}F(w)\right]dw = 0 \quad \text{for} \quad z > 0, \tag{9}$$

where φ and ψ are the functions

$$\left.\begin{aligned}\varphi(wa) &= \pi va\, H_p(va)\, J_p(va)\\ \psi(wa) &= \pi va\, H'_p(va)\, J'_p(va)\end{aligned}\right\}. \tag{10}$$

We shall show how the functions F(w) and G(w) are to be found so that they satisfy the system of functional equations (6)-(9) to which our problem is reduced. The solution of this problem has a more complex form than for the case of symmetric waves (ref. 1) because we have only one unknown function in the latter case.

For the sake of generality we assume that the tube with the open end contains two waves: one is an electric wave with amplitude A and wave number -h while the other is a magnetic wave with amplitude B and wave number $-\tilde{h}$. We draw the contour C in such a manner that it covers the points w=-h and $w=-\tilde{h}$ from below with a series of infinitely narrow loops. For the sake of definiteness we assume that Imk>0 in all of the discussions; we approach the limit Imk=0 only in the final equations.

1. Relationship (6) will be satisfied if the function G(w) in the lower semiplane (i.e., when Imv≤0) is holomorphic everywhere with the exception of point $w=-\tilde{h}$ where it has a simple pole with a residue B and when $|w| \to \infty$ in this semiplane the function tends uniformly to zero.

2. If the function G(w) has the properties specified above, then, in order that relationships (7) be satisfied, the function F(w) must be holomorphic everywhere in the lower semiplane Imw≤0 with the exception of the point w=-h, where it must have a simple pole with residue A, and at the point w=-k where it must also have a simple pole. Furthermore, the product wF(w) must tend uniformly to zero at infinity in the lower semiplane.

3. Relationship (8) will be satisfied if the product $v\varphi(wa)F)w)$ is holomorphic in the upper semiplane (when Imw≥0) and if it tends uniformly zero in this semiplane when $|w| \to \infty$.

4. If the product $v\varphi(wa)F(w)$ has the above properties, then in order for relationship (9) to be satisfied it is sufficient that the product $\frac{\psi(wa)}{v}G(w)$ be holomorphic in the upper semiplane everywhere, with the exception of the point w=k where it has a simple pole. Furthermore, this product must tend uniformly to zero at infinity in the upper semiplane.

The considerations presented above compel us to seek the functions F(w) and G(w) in the form

$$\left.\begin{aligned} F(w) &= \frac{1}{\sqrt{k-w}\,\varphi_2(wa)}\left(\frac{F_1}{w+h}+\frac{F_2}{k+w}\right) \\ G(w) &= \frac{\sqrt{k-w}}{\psi_2(wa)}\left(\frac{G_1}{w+\tilde{h}}+\frac{G_2}{k-w}\right) \end{aligned}\right\}, \tag{11}$$

where $\varphi_2(wa)$ and $\psi_2(wa)$ are functions which are contained in the expansion of /331

functions (10) into factors (see equations (17) below) and are holomorphic in themselves in the lower semiplane Imw≤0.

Expressions (5) for the surface current density must have the following form when z>0:

$$\left.\begin{aligned} j_\varphi &= \cos(p\varphi+\varphi_0)\left[Be^{-i\tilde{h}z}+\ldots\right] \\ j_z &= \sin(p\varphi+\varphi_0)\left[Ae^{-ihz}-\frac{ip}{a}\frac{\tilde{h}}{k^2-\tilde{h}^2}Be^{-i\tilde{h}z}+\ldots\right] \end{aligned}\right\}, \tag{5a}$$

where the terms which have been written out correspond to the incident waves while the sequence of dots replace terms which characterize the current occurring as a result of diffraction at the end. Therefore the constants F_1 and $\tilde{G}_1$ in (11) are associated with the amplitudes A and B of current density by the following relationships:

$$F_1 = \frac{A}{2\pi i}\sqrt{k+h}\,\varphi_1(ha) \qquad G_1 = \frac{B}{2\pi i}\frac{\psi_1(\bar{h}a)}{\sqrt{k+\bar{h}}}, \tag{12}$$

where

$$\varphi_1(wa) = \varphi_2(-wa) \qquad \psi_1(wa) = \psi_2(-wa)$$

are functions which are also contained in expansion (17).

The constants F_2 and G_2 are determined from the requirement that the function under the integral sign in (7) have no pole at w=-k, while the function under the integral sign in (9) has no pole at w=k. This gives us two algebraic equations

$$F_2 - k\Delta\left(-\frac{2k}{k-\bar{h}}G_1 + G_2\right) = 0,$$

$$kG_2 + \Delta\left(\frac{2k}{k+h}F_1 + F_2\right) = 0,$$

where

$$\Delta = \frac{ip}{2ka}\frac{\varphi_1(ka)}{\psi_1(ka)}. \tag{13}$$

From this we obtain the constants F_2 and G_2:

$$\left.\begin{aligned} F_2 &= -\frac{\Delta^2\frac{2k}{k+h}F_1 + \Delta\frac{2k^2}{k-\bar{h}}G_1}{1+\Delta^2} \\ G_2 &= -\frac{\Delta\frac{2}{k+h}F_1 - \Delta^2\frac{2k}{k-\bar{h}}G_1}{1+\Delta^2} \end{aligned}\right\}. \tag{14}$$

Equations (11)-(14) give us the desired solution to the system of functional equations (6)-(9) and thus the solution to the formulated electrodynamic problem.

In the future we shall consider only those cases where there is only one wave traveling inside the waveguide towards the open end--an electric wave or /332 a magnetic wave. Therefore we shall assume that one of the constants F_1 or G_1 (i.e., A or B) is alternately equal to zero. In this case according to equations (14) both of the constants F_2 and G_2 always differ from zero (for asymmetric waves) so that the complex electromagnetic field, which occurs during the diffraction of asymmetric electric or magnetic waves at the open end of the waveguide, will always have an electric as well as a magnetic Hertz function. Therefore, among waves which are traveling from the end of the tube into its interior we have all of the electric and magnetic waves with the same azimuthal relationship as the incident wave. Symmetric waves for which p=0 serve as an exception in this respect because for these Δ=0 and F_2=G_2=0, so that the electromagnetic field has only the Hertz function.

The occurrence of the other Hertz function, which is absent from the field of the incident wave, is associated, from the mathematical point of view, with the fact that the use of only one function F(w) or G(w) makes it impossible to satisfy all of the functional equations (6)-(9), i.e., to satisfy all of the boundary values of the problem (including the condition $j_z=0$ at the edge of the wall.

Section 2. The Properties of the Auxiliary Functions

Let us introduce a dimensionless parameter

$$\varkappa = ka = \frac{2\pi a}{\lambda} \tag{15}$$

and dimensionless variables

$$w' = wa \qquad v' = va = \sqrt{\varkappa^2 - (wa)^2}, \tag{16}$$

where in the confines of the present section we shall simply write w, v in place of w', v' for the sake of brevity.

The breakdown of functions (10)

$$\left.\begin{aligned} \varphi(w) &= \pi v H_p(v) J_p(v) = \varphi_1(w)\varphi_2(w) \\ \psi(w) &= \pi v H_p'(v) J_p'(v) = \psi_1(w)\psi_2(w) \end{aligned}\right\} \tag{17}$$

into factors such that $\varphi_1(w)$ and $\psi_1(w)$ are holomorphic and contain no zeros in the upper semiplane $\mathrm{Im}\,w \geq 0$ while $\varphi_2(w)$ and $\psi_2(w)$ have the same properties in the lower semiplane $\mathrm{Im}\,w \leq 0$, is obtained by means of equations

$$\chi_1(u) = \frac{1}{2\pi i}\int_{-i\varkappa_0-\infty}^{-i\varkappa_0+\infty} \frac{\chi(w)\,dw}{w-u} \qquad \chi_2(u) = -\frac{1}{2\pi i}\int_{i\varkappa_0-\infty}^{i\varkappa_0+\infty} \frac{\chi(w)\,dw}{w-u},$$

where $0<\varkappa_0<\mathrm{Im}\,\varkappa$. For the function $\varphi(w)$

$$\chi(w) = \ln\varphi(w) \qquad \varphi_1(w) = e^{\chi_1(w)} \qquad \varphi_2(w) = e^{\chi_2(w)}, \tag{18}$$

while for the function $\psi(w)$

$$\chi(w) = \ln\psi(w) \qquad \psi_1(w) = e^{\chi_1(w)} \qquad \psi_2(w) = e^{\chi_2(w)}. \tag{19}$$

In this case

$$\varphi_1(w) = \varphi_2(-w) \qquad \psi_1(w) = \psi_2(-w). \tag{20}$$

Such breakdown equations were used frequently in the past. We designate /333
ν_m (m=1, 2, ...) the m-th root of equation

$$J_p(\nu) = 0,$$

and by μ_m (m=1, 2, ...) the m-th root of equation

$$J'_p(\mu)=0,$$

where the positive numbers ν_m and μ_m are placed in order of increasing magnitude (generally speaking we should write $\nu_m^{(p)}$ and $\mu_m^{(p)}$ but in order to simplify matters we drop the superscript p and consider it to be fixed). For p=1, 2, ... the relative order of the roots ν_m and μ_m is as follows:

$$\mu_1<\nu_1<\mu_2<\nu_2<\ldots<\mu_m<\nu_m<\mu_{m+1}<\ldots$$

The function $\varphi(w)$ becomes equal to zero at the points $\pm\gamma_m$ where

$$\gamma_m=\sqrt{\varkappa^2-\nu_m^2}\quad (\operatorname{Im}\gamma_m>0 \quad \text{for} \quad \operatorname{Im}\varkappa>0) \tag{21}$$

is the dimensionless wave number of wave E_{pm}, while the function $\psi(w)$ becomes equal to zero at the points $\pm\tilde{\gamma}_m$ where

$$\tilde{\gamma}_m=\sqrt{\varkappa^2-\mu_m^2}\quad (\operatorname{Im}\tilde{\gamma}_m>0 \quad \text{for} \quad \operatorname{Im}\varkappa>0) \tag{22}$$

is the dimensionless wave number of wave H_{pm}. The usual wave numbers w_m (for E_{pm}) and $\tilde{w}_m$ (for H_{pm}) are given by the equations

$$w_m=\frac{\gamma_m}{a}=\sqrt{k^2-\left(\frac{\nu_m}{a}\right)^2}\qquad \tilde{w}_m=\frac{\tilde{\gamma}_m}{a}=\sqrt{k^2-\left(\frac{\mu_m}{a}\right)^2}. \tag{23}$$

In the final expressions we usually consider the parameter $\varkappa$ as a real positive number and are concerned with the values of the functions for positive values of w within the limits

$$-\varkappa<w<\varkappa. \tag{24}$$

Using these values of w it is expedient to transform the initial equations for the functions φ_1, φ_2, ψ_1, ψ_2 in the same manner as carried out earlier for symmetric waves E_0 and H_0 (ref. 1). As a result we obtain the expressions

$$\varphi_1(w)=\varphi_2(-w)=\sqrt{\pi(\varkappa+w)H_p(v)J_p(v)}\prod_{m=1}^{n}\frac{\gamma_m+w}{\gamma_m-w}e^{\frac{1}{2}S(w)} \tag{25}$$

$$\psi_1(w)=\psi_2(-w)=\sqrt{\pi(\varkappa-w)H'_p(v)J'_p(v)}\prod_{m=1}^{n}\frac{\tilde{\gamma}_m+w}{\tilde{\gamma}_m-w}e^{\frac{1}{2}\tilde{S}(w)} \tag{26}$$

Here n designates the number of waves E_{pm}, which are capable of propagating in the waveguide for a given $\varkappa$, while $\tilde{n}$ is the number of propagating waves H_{pm}. In other words we assume that

$$\nu_n < \varkappa < \nu_{n+1} \qquad \mu_{\tilde{n}} < \varkappa < \mu_{\tilde{n}+1}.$$

We let

$$S(w) = X(w) + iY(w). \tag{27}$$

The real part of the function S is equal to (the integral for X is taken in the principal value) /334

$$\left.\begin{aligned} X(w_0) &= \frac{1}{\pi}\int\limits_{-\varkappa}^{\varkappa} \frac{\Omega(v)\,dw}{w - w_0} = -w_0 E(v_0, \varkappa) \\ E(v_0, \varkappa) &= \frac{2}{\pi}\int\limits_{0}^{\varkappa} \frac{\Omega(v)\,v - \Omega(v_0)\,v_0}{v^2 - v_0^2}\,\frac{dv}{\sqrt{\varkappa^2 - v^2}} \end{aligned}\right\}, \tag{28}$$

where $v_0 = \sqrt{\varkappa^2 - w_0^2}$. The imaginary part of S when $0 \leq w_0 \leq \varkappa$ is given by the equation

$$Y(w_0) = \frac{2w_0}{\pi} - \Omega(v_0) + 2\lim_{N\to\infty}\left\{-\sum_{m=n+1}^{N} \arcsin\frac{w_0}{\sqrt{\nu_m^2 - v_0^2}} + \frac{w_0}{\pi}\int\limits_{\varkappa}^{\nu_N} \frac{\Omega(v)\,v}{v^2 - v_0^2}\,\frac{dv}{\sqrt{v^2 - \varkappa^2}}\right\}. \tag{29}$$

The function Ω contained in the above expressions is determined from the relationships

$$\left.\begin{aligned} \Omega(v) &= \arg H_p(v) + \frac{\pi}{2} = \operatorname{arctg}\frac{N_p(v)}{J_p(v)} + \frac{\pi}{2} \\ \Omega(0) &= 0 \qquad \Omega(\nu_m) = m\pi \end{aligned}\right\}. \tag{30}$$

If, in equations (28) and (29), we replace the function Ω by the function $\tilde{\Omega}$ which is given by the equations

$$\left.\begin{aligned} \tilde{\Omega}(v) &= \arg H_p'(v) - \frac{\pi}{2} = \operatorname{arc\,tg}\frac{N_p'(v)}{J_p'(v)} - \frac{\pi}{2} \\ \tilde{\Omega}(0) &= 0 \qquad \tilde{\Omega}(\mu_m) = (m-1)\pi \end{aligned}\right\}, \tag{31}$$

and also replace ν_m by μ_m and n by $\tilde{n}$, we obtain the real and imaginary parts of the function

$$\tilde{S}(w) = \tilde{X}(w) + i\tilde{Y}(w). \tag{32}$$

We note that when $v \gg 1$ (or when $v \gg p^2$) the following asymptotic equations exist for $\Omega(v)$ and $\tilde{\Omega}(v)$

$$\left.\begin{aligned}\Omega(v)&=v-\frac{2p-1}{4}\pi+\frac{p^2-\frac{1}{4}}{2v}\\ \tilde{\Omega}(v)&=v-\frac{2p+1}{4}\pi+\frac{p^2+\frac{3}{4}}{2v}\end{aligned}\right\}, \tag{33}$$

from which we obtain the approximate equations for the higher order ν_m and μ_m roots

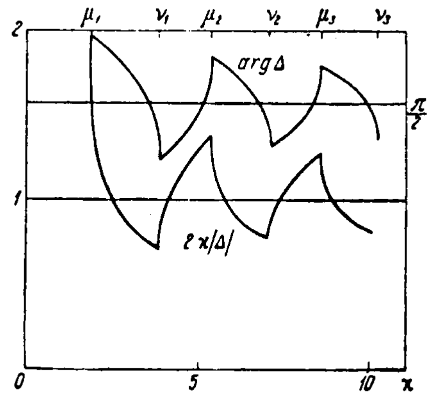
$$\nu_m=\pi\left[m+\frac{2p-1}{4}-\frac{p^2-\frac{1}{4}}{2\pi^2\left(m+\frac{2p-1}{4}\right)}\right],$$

$$\nu_{m+1}=\pi\left[m+\frac{2p+1}{4}-\frac{p^2+\frac{3}{4}}{2\pi^2\left(m+\frac{2p+1}{4}\right)}\right].$$

The above equations make it possible to compute the functions φ_1, φ_2, ψ_1 /335

and ψ_2 which are contained in the general solution of the problem on the electromagnetic waves in an open tube, which has been presented above in section 1.

In the figure we present (for p=1) the absolute value of the function Δ (13), multiplied by $2\varkappa$, and its phases arg Δ as a function of the parameter $\varkappa$=ka. These curves oscillate respectively near the values p=1 and $\pi/2$, and undergo breaks at the "critical" values of the parameter $\varkappa$ when one of the waves E_{pm} or H_{pm} changes from a damped wave to a propagating wave; the amplitude of these oscillations drops off slowly as $\varkappa$ increases. This unique "resonance" behavior is also common to other auxiliary functions, and leads to characteristic breaks in the curves which show the frequency dependence of various physical quantities for a waveguide with an open end.



Section 3. Current Density at the Wall. Coefficients of Reflection and Transformation

If an electric wave E_{pl} which propagates through the tube with wave number $-h=-w_l$ moves into the open end of the tube, then, according to equations (11)-(14) we have

$$\left.\begin{aligned}F(w)&=\frac{A}{2\pi i}\frac{\sqrt{k+h}\,\varphi_1(\gamma_l)}{\sqrt{k-w}\,\varphi_2(wa)}\left[\frac{1}{w+h}-\frac{\Delta^2}{1+\Delta^2}\frac{2k}{k+h}\frac{1}{k+w}\right]\\ G(w)&=\frac{A}{2\pi i}\frac{-2\Delta}{1+\Delta^2}\frac{\varphi_1(\gamma_l)}{\sqrt{k+h}\sqrt{k-w}\,\psi_2(wa)}\end{aligned}\right\}. \tag{34}$$

This makes it possible for us to obtain from equation (5), when z>0, the components of the surface current density on the wall of the tube, in the form

$$\left.\begin{aligned} j_z &= \sin(p\varphi + \varphi_0) A \left(e^{-ihz} + \sum_{m=1}^{\infty} R_{l,m} e^{iw_m z} + ip \sum_{m=1}^{\infty} \frac{\gamma_m}{\mu_m^2} T_{l,m} e^{i\tilde{w}_m z} + Q_z \right) \\ j &= \cos(p\varphi + \varphi_0) A \left(\sum_{m=1}^{\infty} T_{l,m} e^{i\tilde{w}_m z} + Q_\varphi \right). \end{aligned}\right\} \tag{35}$$

Here

$$R_{l,m} = \sqrt{\frac{\varkappa + \gamma_l}{\varkappa - \gamma_m}} \frac{\varphi_1(\gamma_l)}{(\gamma_l + \gamma_m)\varphi_2'(\gamma_m)} \left[1 - \frac{\Delta^2}{1+\Delta^2} \frac{2\varkappa(\gamma_l + \gamma_m)}{(\varkappa + \gamma_l)(\varkappa + \tilde{\gamma}_m)}\right] \tag{36}$$

is the transformation coefficient (for the current) of wave E_{pl} into wave E_{pm} /336

(in this case R_{ll} is simply the reflection coefficient of the wave E_{pl} from the open end), while

$$T_{l,m} = \frac{-2\Delta}{1+\Delta^2} \frac{\varphi_1(\gamma_l)}{\sqrt{(\varkappa + \gamma_l)(\varkappa - \gamma_m)}\, \psi_2'(\tilde{\gamma}_m)} \tag{37}$$

will be called the current transformation coefficient of the wave E_{pl} into wave H_{pm}.

Let us now consider other possibilities when a magnetic wave H_{pl} with current amplitude B and wave number $-\tilde{h}=-\tilde{w}_l$ arrives at the open end of the tube. In this case

$$\left.\begin{aligned} F(w) &= \frac{B}{2\pi i} \frac{-2\Delta}{1+\Delta^2} \frac{k^2 \psi_1(\tilde{\gamma}_l)}{(k-\tilde{h})\sqrt{k+\tilde{h}}\,(k+w)\sqrt{k-w}\,\varphi_2(wa)} \\ G(w) &= \frac{B}{2\pi i} \frac{\sqrt{k-w}\,\psi_1(\gamma_l)}{\sqrt{k+\tilde{h}}\,\psi_2(wa)} \left(\frac{1}{w+\tilde{h}} + \frac{\Delta^2}{1+\Delta^2} \frac{2k}{k-h} \frac{1}{k-w}\right) \end{aligned}\right\} \tag{38}$$

and the surface current density components, when z>0, are equal to

$$\left.\begin{aligned} j_\varphi &= \cos(p\varphi + \varphi_0) B \left(e^{-i\tilde{h}z} + \sum_{m=1}^{\infty} \tilde{R}_{l,m} e^{i\tilde{w}_m z} + \tilde{Q}_\varphi \right) \\ j_z &= \sin(p\varphi + \varphi_0) B \left[ip \left(-\frac{\gamma_l}{\mu_l^2} e^{-i\tilde{h}z} + \sum_{m=1}^{\infty} \frac{\gamma_m}{\mu_m^2} \tilde{R}_{l,m} e^{i\tilde{w}_m z} \right) + \right. \\ &\left. + \sum_{m=1}^{\infty} \tilde{T}_{l,m} e^{iw_m z} + \tilde{Q}_z \right] \end{aligned}\right\}. \tag{39}$$

Here $\tilde{R}_{lm}$ is the current transformation coefficient of the wave H_{pl} into wave H_{pm} and is given by the expression

$$\tilde{R}_{l,m} = \sqrt{\frac{\varkappa - \tilde{\gamma}_m}{\varkappa + \gamma_l}} \frac{\psi_1(\tilde{\gamma}_l)}{(\gamma_l + \tilde{\gamma}_m)\,\psi_2'(\gamma_m)} \left[1 + \frac{\Delta^2}{1 + \Delta^2} \frac{2\varkappa(\tilde{\gamma}_l + \tilde{\gamma}_m)}{(\varkappa - \tilde{\gamma}_l)(\varkappa - \tilde{\gamma}_m)}\right] \tag{40}$$

where $\tilde{R}_{l,l}$ is simply the reflection coefficient for the wave H_{pl} from the end of the tube. It is natural to call the quantity

$$\tilde{T}_{l,m} = \frac{-2\Delta}{1 + \Delta^2} \frac{\varkappa^2 \psi_1(\tilde{\gamma}_l)}{(\varkappa - \tilde{\gamma}_l)\sqrt{\varkappa + \tilde{\gamma}_l}\,(\varkappa + \gamma_m)\sqrt{\varkappa - \gamma_m}\,\varphi_2'(\gamma_m)} \tag{41}$$

the current transformation coefficient for wave H_{pl} into wave E_{pm}.

The coefficients $R_{l,m}$ and $\tilde{T}_{l,m}$ give complex amplitudes for the longitudinal component of the current density in electric waves which travel from the open end of the tube and which are excited by the incident wave, while the coefficients $T_{l,m}$ and $\tilde{R}_{l,m}$ determine the complex amplitudes of the azimuthal components of the current density for magnetic waves which move from the end of the tube into its interior.

We note that for electric waves the currents flow in the longitudinal direction only while the magnetic waves have both a longitudinal and an azimuthal component of current density. Near the critical frequency of a given magnetic wave the aximuthal component for the current density exceeds the longitudinal component. However, as the frequency is increased they become /337
comparable and at frequencies which substantially exceed the critical frequency, the longitudinal component becomes the dominant one. Let us introduce the coefficients

$$\rho_{l,m} = R_{l,m} \qquad \tau_{l,m} = ip \frac{\gamma_m}{\mu_m^2} T_{l,m} \tag{42}$$

and

$$\tilde{\rho}_{l,m} = -\frac{\mu_l^2}{\mu_m^2} \frac{\gamma_m}{\tilde{\gamma}_l} \tilde{R}_{l,m} \qquad \tilde{\tau}_{l,m} = -\frac{1}{ip} \frac{\mu_l^2}{\tilde{\gamma}_l} \tilde{T}_{l,m}, \tag{43}$$

which, as we can easily see from equations (35) and (39), are the reflection and transformation coefficients for various waves along the longitudinal current component. It is expedient to utilize these quantities for large values of the parameter $\varkappa$ when they all have the same dependence on $\varkappa$.

The reflection and transformation coefficients determine the electromagnetic field inside the waveguide. Usually we are interested only in the propagating waves which carry back part of the energy introduced by the incident wave, into the tube and which determine the field inside the tube at sufficiently great distances from the end where damped waves can be neglected. The best illustration of the transformation of one wave into another is given by the power transformation coefficients. If, for example, we designate by (E_{pl}, H_{pm}) that portion of the energy in the incident wave E_{pl} which is carried away by the inverse wave H_{pm}, we shall have the following expression for the power transformation coefficients;

$$\left.\begin{array}{ll}(E_{pl}, E_{pm}) = \frac{\gamma_m}{\gamma_l}|R_{l,m}|^2 & (E_{pl}, H_{pm}) = \frac{\gamma_m}{\gamma_l}\frac{\varkappa^2}{\mu_m^2}\left(1-\frac{p^2}{\mu_m^2}\right)|T_{l,m}|^2 \\ (H_{pl}, H_{pm}) = \frac{\tilde{\gamma}_m}{\tilde{\gamma}_l}\frac{\mu_l^2}{\mu_m^4}\frac{1-\frac{p^2}{\mu_m^2}}{1-\frac{p^2}{\mu_l^2}}|\tilde{R}_{l,m}|^2 & (H_{pl}, E_{pm}) = \frac{\gamma_m}{\tilde{\gamma}_l}\frac{\mu^2}{\varkappa^2}\frac{1}{1-\frac{p^2}{\mu_l^2}}|\tilde{T}_{l,m}|^2\end{array}\right\}. \qquad (44)$$

From the general equations presented above for the current coefficient of reflection and transformation, it follows that for propagating waves

$$(E_{pl}, E_{pm}) = (E_{pm}, E_{pl}); \quad (H_{pl}, H_{pm}) = (H_{pm}, H_{pl}); \quad (E_{pl} H_{pm}) = (H_{pm}, E_{pl})$$

and

$$\arg R_{l,m} = \arg R_{m,l}; \; \arg \tilde{R}_{l,m} = \arg \tilde{R}_{m,l}; \quad \arg T_{l,m} = \arg \tilde{T}_{m,l}.$$

These symmetry relationships must be satisfied in the general case for any discontinuity in the waveguide.

The terms Q_φ, Q_z, $\tilde{Q}_\varphi$, $\tilde{Q}_z$ in equations (35) and (39) are expressed in terms of integrals along a loop which covers the cut $k \to k+i\infty$. For large values of z these integrals decrease in inverse proportion to some power of z. It is easy to show that they produce the surface current density on the external side of the waveguide's wall whereas the remaining terms represent the current density on the internal side of the wall directly associated with waves inside the tube.

Section 4. The Huygens Principle

The most rational approach for computing the radiation in accordance with the Huygens principle consists of using an approximate expression which associates the radiation with the electromagnetic field of the wave incident at the open end. Using this method the radiation field of the electric wave E_{pl} can be obtained in the form /338

$$\left.\begin{array}{c}E_\vartheta = H_\varphi = -\sin(p\varphi+\varphi_0)(-i)^p \frac{2\pi a}{c} A \frac{\sin\vartheta\, J_p(\varkappa \sin\vartheta)}{\cos\vartheta - \cos\vartheta_l}\frac{e^{ikR}}{R} \\ E_\varphi = -H_\vartheta = 0\end{array}\right\}, \qquad (45)$$

where A is the amplitude of the longitudinal component for the current density of wave $E_{p\iota}$, while ϑ_ι is the angle which is associated with the wave number $-h=-w_\iota$ of this wave by the relationship

$$k\cos\vartheta_\iota = -h. \tag{46}$$

By using this method we obtain equations for the magnetic wave which are more complicated. Specifically for the wave $H_{p\iota}$ in the wave zone we have the field

$$\left.\begin{aligned} E_\vartheta = H_\varphi &= -\sin(p\varphi+\varphi_0)(-i)^{p+1}\frac{2\pi a}{c}B\frac{p\varkappa}{\mu_\iota^2}(1+\cos\tilde{\vartheta}_\iota\cos\vartheta)\frac{J_p(\varkappa\sin\vartheta)}{\sin\vartheta}\frac{e^{ikR}}{R} \\ E_\varphi = -H_\vartheta &= -\cos(p\varphi+\varphi_0)(-i)^{p+1}\frac{2\pi a}{c}B\frac{J'_p(\varkappa\sin\vartheta)}{\cos\vartheta-\cos\tilde{\vartheta}_\iota}\frac{e^{ikR}}{R} \end{aligned}\right\}, \tag{47}$$

where B designates the amplitude of the azimuthal current density component of wave $H_{p\iota}$ (see equation (52)), while the angle $\tilde{\vartheta}_\iota$ is associated with its wave number $-\tilde{h}=-\tilde{w}_\iota$ by the relationship

$$k\cos\tilde{\vartheta}_\iota = -\tilde{h}. \tag{48}$$

Here we have introduced spherical coordinates R, φ, ϑ with the same origin as that of the coordinate system r, φ, z; $\vartheta=0$ is the direction of the positive z axis (along the tube) while $\vartheta=\pi$ corresponds to the extension of the waveguide.

There are other possible methods for the approximate computation of the radiation field, which are based on the application of the Kirchhoff equations to the auxiliary quantities which characterize the electromagnetic field. By using the Hertz electric vector we obtain exactly the same equation (45) for the electric wave as before. On the other hand if we apply the Kirchhoff equations to the Hertz magnetic vector we obtain a radiation field for the wave $H_{p\iota}$ in the form

$$\left.\begin{aligned} E_\vartheta = H_\varphi &= 0 \\ E_\varphi = -H_\vartheta &= -\cos(p\varphi+\varphi_0)(-i)^{p+1}\frac{2\pi a}{c}B\frac{\sin^2\vartheta}{\sin^2\vartheta_\iota}\frac{J'_p(\varkappa\sin\vartheta)}{\cos\vartheta-\cos\vartheta_\iota}\frac{e^{ikR}}{R} \end{aligned}\right\}. \tag{49}$$

In comparing the two calculation methods for symmetric waves (ref. 1, section 4), we note that the first method is to be preferred from considerations of uniqueness; a comparison of the radiation characteristics for wave H_{01} carried out at the same time (ref. 1, fig. 7) also shows a preference for the first method. For the asymmetric waves its advantage is obvious. In the first place equation (49) for the radiation characteristics of wave $H_{1\iota}$ shows a dip when $\vartheta=\pi$ which does not correspond to the true state of things. In the second place, under the condition $\varkappa\gg1$ when only good results can be expected from the Huygens principle, the field $E_\vartheta=H_\varphi$ according to equation (47), is much stronger than the field $E_\varphi = -H_\vartheta$ and this result is in good agreement with the rigorous theory. However, according to equation (49) the fields $E_\vartheta = H_\varphi = 0$.

These results require that we reject the Huygens principle in other problems when this principle is formulated in the form which utilizes the auxiliary potentials of electromagnetic wave. This form is superfluous when it leads to the same results as the Huygens principle for fields (first method) and if it produces other results then these are completely unreliable. Therefore, in the future we shall interpret the Huygens principle to mean the first method of calculation. Below we compare the Huygens principle with the exact solution. /339

We should like to point out another method for carrying out the approximate calculations which leads to the same final equations as the Huygens principle (for fields), but which has certain methodological advantages in several cases. Specifically we shall assume that the current density at the wall is the same as at the wave which propagates inside an infinite tube (i.e., when z>0, the current density is described by the terms of equation (5a) which have been written out) while along the extension of the wall (i.e., for z < 0, it is equal to zero). The functions F(w) and G(w) which give this current distribution are as follows

$$F(w) = \frac{A}{2\pi i}\,\frac{1}{w+h} \qquad\qquad G(w) = 0$$

for the wave $E_{p\,l}$, and

$$F(w) = -\frac{B}{2\pi i}\,\frac{ipa}{\mu_l^2}\,\frac{k^2 - \tilde{h}w}{v^2} \qquad\qquad G(w) = \frac{B}{2\pi i}\,\frac{1}{w+\tilde{h}}$$

for the wave $H_{p\,l}$.

From these the radiation field can be easily determined. Substituting functions F and G into equations (51) and (52) of the next section we again obtain equations (45) and (47) of the Huygens principle. The equivalence of both methods (not only for waveguides but for other radiating systems) may be easily proven in the general form.

Section 5. The Radiation Characteristics

In the wave zone, more precisely outside the waveguide at distances R from its end such that

$$kR \gg 1\,, \qquad kR\sin^2\vartheta \gg 1 \tag{50}$$

we obtain the following equations for the Hertz functions from the exact expressions (ref. 4) by using the method of steepest descent:

$$\left.\begin{aligned} \Pi &= -(-i)^{p+1}\,\frac{4\pi^2 a}{ck}\,J_p(\varkappa\sin\vartheta)\,F(k\cos\vartheta)\,\frac{e^{ikR}}{R} \\ \tilde{\Pi} &= -(-i)^{p}\,\frac{4\pi^2 a}{ck}\,J_p'(\varkappa\sin\vartheta)\,\frac{G(k\cos\vartheta)}{\sin\vartheta}\,\frac{e^{ikR}}{R} \end{aligned}\right\}. \tag{51}$$

From the open end we have diverging spherical waves whose electromagnetic fields are equal to

$$\left.\begin{aligned} E_\vartheta = H_\varphi &= -\sin(p\varphi + \varphi_0)\, k^2 \sin\vartheta\, \Pi \\ E_\varphi = -H_\vartheta &= \cos(p\varphi + \varphi_0)\, k^2 \sin\vartheta\, \tilde{\Pi} \end{aligned}\right\}. \tag{52}$$

The directional propagation of radiation is given by the function

$$\Sigma(\vartheta, \varphi) = \frac{c}{8\pi}(E_\vartheta H_\varphi^* - E_\varphi H_\vartheta^*) R^2 = \frac{c}{8\pi}(|E_\varphi|^2 + |H_\varphi|^2) R^2,$$

so that

$$\Sigma(\vartheta, \varphi)\, dO$$

is the power radiated inside an elementary solid angle dO. It follows from equation (52) that

$$\Sigma(\vartheta, \varphi) = \sin^2(p\varphi + \varphi_0)\,\sigma(\vartheta) + \cos^2(p\varphi + \varphi_0)\,\tilde{\sigma}(\vartheta), \tag{53}$$

where the positive functions $\sigma(\vartheta)$ and $\tilde{\sigma}(\vartheta)$ depend only on the angle ϑ. /340
Because

$$\left.\begin{aligned} \sigma(\vartheta) &= \frac{c}{8\pi} k^4 \sin^2\vartheta\, |\Pi|^2 R^2 \\ \tilde{\sigma}(\vartheta) &= \frac{c}{8\pi} k^4 \sin^2\vartheta\, |\tilde{\Pi}|^2 R^2 \end{aligned}\right\}, \tag{54}$$

$\sigma(\vartheta)$ may be called the electric radiation characteristic while $\tilde{\sigma}(\vartheta)$ may be called the magnetic radiation characteristics. When $\vartheta=0$ and $\vartheta=\pi$ the function $\Sigma(\vartheta, \varphi)$ must not depend on φ; therefore, the following relationship exists between the electric and magnetic characteristics

$$\Sigma(0, \varphi) = \tilde{\sigma}(0) = \sigma(0) \qquad \Sigma(\pi, \varphi) = \sigma(\pi) = \tilde{\sigma}(\pi). \tag{55}$$

Relationships (51)-(55) may be applied to various approximate calculations. We shall distinguish the radiation characteristics according to the Huygens principle from the exact values by the subscript 1. From equation (45) we obtain

$$\left.\begin{aligned} \sigma_1(\vartheta) &= P \frac{\varkappa}{2\pi\gamma_l}\left[\frac{\sin\vartheta\, J_p(\varkappa \sin\vartheta)}{\cos\vartheta - \cos\vartheta_l}\right]^2 \\ \tilde{\sigma}_1(\vartheta) &= 0 \end{aligned}\right\}, \tag{56}$$

where

$$P = \frac{\pi^2 a^2}{c} \frac{\gamma_l}{\varkappa} |A|^2 \tag{57}$$

is the power of the wave E_{pl} (p=1, 2, ...). In the same way, from equation (47) for the wave $H_{p\nu}$ we have

$$\left.\begin{aligned} \sigma_1(\vartheta) &= P\frac{p^2\varkappa}{2\pi\tilde{\gamma}_l\mu_l^2\left(1-\frac{p^2}{\mu_l^2}\right)}\left[\left(1+\cos\tilde{\vartheta}_l\cos\vartheta\right)\frac{J_p(\varkappa\sin\vartheta)}{\sin\vartheta}\right]^2 \\ \tilde{\sigma}_1(\vartheta) &= P\frac{\mu_l^2}{2\pi\varkappa\tilde{\gamma}_l\left(1-\frac{p^2}{\mu_l^2}\right)}\left[\frac{J_p'(\varkappa\sin\vartheta)}{\cos\vartheta-\cos\tilde{\vartheta}_l}\right]^2 \end{aligned}\right\}, \tag{58}$$

where the power P of wave H_{pl} is equal to

$$P=\frac{\pi^2a^2}{c}\frac{\varkappa\tilde{\gamma}_l}{\mu_l^2}\left(1-\frac{p^2}{\mu_l^2}\right)|B|^2. \tag{59}$$

For the wave H_{1l}

$$\sum\nolimits_1(\pi,\varphi)=\sigma_1(\pi)=\tilde{\sigma}_1(\pi)=P\frac{\varkappa(\varkappa+\tilde{\gamma}_l)^2}{8\pi\tilde{\gamma}_l(\mu_l^2-1)}, \tag{60}$$

while, according to the Huygens principle, for all other waves

$$\sum\nolimits_1(\pi,\varphi)=\sigma_1(\pi)=\tilde{\sigma}_1(\pi)=0. \tag{61}$$

We return to the investigation of the exact expressions for the radiation field. We obtain the radiation of the electric wave from the open end by substituting expressions (34) for functions F and G into equations (51). From this it is now easy to obtain specific design equations. Thus when $\nu_1<\varkappa<\nu_2$ we have the radiation characteristics for the wave E_{pl} in the form

$$\sigma(\vartheta)=P\frac{2\gamma_1(\varkappa+\gamma_1)^2}{\pi^2\nu_1^2\varkappa}\frac{J_p(\varkappa\sin\vartheta)}{\sin^2\vartheta\,|H_p(\varkappa\sin\vartheta)|}\frac{e^{X(\gamma_1)+X(\varkappa\cos\vartheta)}}{\cos^2\vartheta-\cos^2\vartheta_1}\Bigg|1+\cos\vartheta-\frac{2\Delta^2}{1+\Delta^2}\frac{\cos\vartheta-\cos\vartheta_1}{1-\cos\vartheta_1}\Bigg|^2, \tag{62}$$

where, for $\nu_1<\varkappa<\mu_2$ /341

$$\sigma(\vartheta)=P\frac{8\varkappa\gamma_1}{\pi^2\nu_1^2}\frac{|\Delta|^2}{|1+\Delta^2|^2}\frac{J_p'(\varkappa\sin\vartheta)}{\sin^2\vartheta\,|H_p'(\varkappa\sin\vartheta)|}\frac{\cos\vartheta-\cos\tilde{\vartheta}_1}{\cos\vartheta+\cos\tilde{\vartheta}_1}e^{X(\gamma_1)+\tilde{X}(\varkappa\cos\vartheta)}, \tag{63}$$

while for $\mu_2<\varkappa<\nu_2$ the last expression must be multiplied by

$$\frac{\cos\vartheta-\cos\tilde{\vartheta}_2}{\cos\vartheta+\cos\tilde{\vartheta}_2}. \tag{63a}$$

For any value of the parameter $\varkappa$ the radiation characteristics of the E_{pl} wave satisfy the relationships

$$\left.\begin{aligned} &\sigma(\vartheta_l)=\sigma_1(\vartheta_l)=P\frac{\varkappa\gamma_l}{2\pi}J_{p-1}^2(\nu_l) \\ &\sigma(\vartheta_m)=\sigma_1(\vartheta_m)=0 \quad (\text{for } m\neq l) \\ &\tilde{\sigma}(\tilde{\vartheta}_m)=0 \qquad\qquad (m=1,2,\ldots) \end{aligned}\right\}. \tag{64}$$

From expressions (62) and (63) we obtain the energy balance

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sum (\vartheta, \varphi) \sin\vartheta\, d\vartheta = \tag{65}$$
$$= \pi \int_0^{\pi} [\sigma(\vartheta) + \tilde{\sigma}(\vartheta)] \sin\vartheta\, d\vartheta = P \{ 1 - [(E_{p1}, E_{p1}) + (E_{p1}, H_{p1})] \},$$

where (E_{pl}, E_{pl}) and (E_{pl}, H_{pl}) are the reflection and power transformation coefficients introduced above (equations (44)). Equation (65) is suitable when $\nu_1 < \varkappa < \mu_2$ and $\mu_2 < \varkappa < \nu_2$ the function $\tilde{\sigma}(\vartheta)$ has an additional multiplier (63a) which is responsible for the addition of term (E_{pl}, H_{p2}) in equation (65).

If, however, the wave H_{pl} approaches the open end then we obtain a field in the wave zone from equations (51) and (52) by substituting the function F and G from equations (38) into these expressions. We obtain the following magnetic characteristic from the general equations for the wave H_{pl} when $\mu_1 < \varkappa < \mu_2$:

$$\tilde{\sigma}(\vartheta) = P \frac{2\mu_1^2 \gamma_1}{\pi^2 x (\varkappa + \gamma_1)^2} \frac{J_p'(x \sin\vartheta)}{\sin^2\vartheta \, |H_p'(x \sin\vartheta)|} \left| \frac{e^{\tilde{X}(\gamma_1) + X(x\cos\vartheta)}}{\cos^2\vartheta - \cos^2\vartheta_1} \right| 1 - \tag{66}$$
$$- \cos\vartheta + \frac{2\Delta^2}{1+\Delta^2} \frac{\cos\vartheta - \cos\tilde{\vartheta}_1}{1 + \cos\tilde{\vartheta}_1} \Big|^2 .$$

In regard to the electric characteristic, in the case $\mu_1 < \varkappa < \nu_2$, when electric waves with the same azimuthal dependence do not propagate at all, it is equal to

$$\sigma(\vartheta) = P \frac{8x\gamma_1}{\pi^2 \mu_1^2} \frac{|\Delta|^2}{|1+\Delta^2|^2} \frac{J_p(x \sin\vartheta)}{\sin^2\vartheta \, |H_p(x \sin\vartheta)|} e^{\tilde{X}(\gamma_1) + X(x\cos\vartheta)}, \tag{67}$$

and for $\nu_1 < \varkappa < \mu_2$ when the wave E_{pl} is capable of propagation we must add the following factor into expression (67):

$$\frac{\cos\vartheta - \cos\vartheta_1}{\cos\vartheta + \cos\vartheta_1} . \tag{67a}$$

According to equations (66) and (67) the total radiated power is equal to /342

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sum (\vartheta, \varphi) \sin\vartheta\, d\vartheta = \pi \int_0^{\pi} [\sigma(\vartheta) + \tilde{\sigma}(\vartheta)] \sin\vartheta\, d\vartheta = P [1 - (H_{p1}, H_{p1})], \tag{68}$$

where in the case $\nu_1 < \varkappa < \mu_2$ we have the sum $(H_{pl}, H_{pl}) + (H_{pl}, E_{pl})$ instead of (H_{pl}, H_{pl}) in equation (68) due to the multiplier (67a). The radiation characteristics of magnetic waves H_{pl} satisfy the following relationships at all frequencies:

$$\left.\begin{array}{l}\tilde{\sigma}(\tilde{\vartheta}_l)=\tilde{\sigma}_1(\tilde{\vartheta}_l)=P\frac{\varkappa\tilde{\gamma}_l}{2\pi}\left(1-\frac{p^2}{\mu_l^2}\right)J_p^2(\mu_l)\\ \tilde{\sigma}(\tilde{\vartheta}_m)=\tilde{\sigma}_1(\tilde{\vartheta}_m)=0\quad(m\neq l)\\ \sigma(\vartheta_m)=\sigma_1(\vartheta_m)=0\quad(m=1,\ 2,\ \ldots)\end{array}\right\}. \tag{69}$$

We note, in conclusion, that the Hertz functions (51) in the wave zone are associated in the following manner:

$$\lim_{\vartheta\to\pi}\sin\vartheta\cdot\Pi=\lim_{\vartheta\to\pi}\sin\vartheta\cdot\tilde{\Pi}\qquad\lim_{\vartheta\to 0}\sin\vartheta\cdot\Pi=-\lim_{\vartheta\to 0}\sin\vartheta\cdot\tilde{\Pi}. \tag{70}$$

These equalities cause relationships (55) to be satisfied. They show that the formation of a wave field whose radiation is different from zero in the direction $\vartheta=\pi$ is possible only when both the electric and magnetic Hertz functions are present.

Section 6. Approximate Equations

Let us investigate the form which is assumed by the exact expressions, deduced above, when we have the following condition:

$$\varkappa=ka\gg 1, \tag{71}$$

i.e., sufficiently large (compared to wavelength) radiating apertures. We use the equations

$$\left.\begin{array}{lll}\ln\varphi_1(\varkappa\cos\vartheta)=U & \ln\psi_1(\varkappa\cos\vartheta)=\tilde{U} & \text{for}\ \cos\vartheta>0\\ \ln\varphi_1(\varkappa\cos\vartheta)=\ln\varphi(\varkappa\cos\vartheta)+U & \ln\psi_1(\varkappa\cos\vartheta)=\ln\psi(\varkappa\cos\vartheta)+\tilde{U} & \text{for}\ \cos\vartheta<0\end{array}\right\}, \tag{72}$$

(see refs. 1 and 3, section 6), where U and $\tilde{U}$ are the integrals

$$U=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\ln\varphi(\varkappa\sin\tau)\cos\tau d\tau}{\sin\tau-\cos\vartheta}\qquad\tilde{U}=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\ln\psi(\varkappa\sin\tau)\cos\tau d\tau}{\sin\tau-\cos\vartheta} \tag{73}$$

along the contour Γ_0 which passes in the plane of the complex variable τ through the point $\tau=0$ in the direction which provides for the most rapid increase in the real part of function $\cos\tau$. Since the contour Γ_0 is symmetric with respect to the point $\tau=0$, U and $\tilde{U}$ are odd functions of $\cos\vartheta$, and have a discontinuity at $\cos\vartheta=0$. If $\cos\vartheta\neq 0$ and $\varkappa\to\infty$, the functions U and $\tilde{U}$ tend to zero.

By using equations (72) we obtain the following expression for the radiation field of wave E_{pl} in the forward half space ($\frac{\pi}{2}<\vartheta<\pi$):

$$\left.\begin{aligned}
\Pi &= (-i)^p \frac{2\pi a}{ck^2} A \frac{\sin\frac{\vartheta_l}{2}}{\sin\frac{\vartheta}{2}} \frac{J_p(\varkappa \sin\vartheta)}{\cos\vartheta - \cos\vartheta_l} \frac{1 + \Delta^2 \left(\frac{\operatorname{tg}\frac{\vartheta}{2}}{\operatorname{tg}\frac{\vartheta_l}{2}}\right)^2}{1 + \Delta^2} \frac{e^{ikR + U_l + U}}{R} \\
\tilde{\Pi} &= (-i)^{p+1} \frac{2\pi a}{ck^2} A \frac{\Delta}{1 + \Delta^2} \frac{J_p'(\varkappa \sin\vartheta)}{\sin\frac{\vartheta_l}{2} \sin\frac{\vartheta}{2} \sin\vartheta} \frac{e^{ikR + U_l + \tilde{U}}}{R}
\end{aligned}\right\}, \tag{74}$$

while in the rear half space, when $0<\vartheta<\frac{\pi}{2}$, we have /343

$$\left.\begin{aligned}
\Pi &= (-i)^p \frac{2}{ck^3} A \frac{\sin\frac{\vartheta_l}{2}}{\sin\frac{\vartheta}{2}} \frac{1}{\sin\vartheta H_p(\varkappa \sin\vartheta)(\cos\vartheta - \cos\vartheta_l)} \frac{1 + \Delta^2 \left(\frac{\operatorname{tg}\frac{\vartheta}{2}}{\operatorname{tg}\frac{\vartheta_l}{2}}\right)^2}{1 + \Delta^2} \frac{e^{ikR + U_l + U}}{R} \\
\tilde{\Pi} &= (-i)^{p+1} \frac{2}{ck^3} A \frac{\Delta}{1 + \Delta^2} \frac{1}{\sin\frac{\vartheta_l}{2} \sin\frac{\vartheta}{2} \sin^2\vartheta\, H_p'(\varkappa \sin\vartheta)} \frac{e^{ikR + U_l + \tilde{U}}}{R}
\end{aligned}\right\}. \tag{75}$$

Here U_l is the value of the function U when $\vartheta=\pi-\vartheta_l$, i.e., when

$$\cos\vartheta = \frac{\gamma_l}{\varkappa}$$

(see equation (46)), while the quantity Δ (13) is equal to

$$\Delta = \frac{ip}{2\varkappa} e^{U_0 - \tilde{U}_0}. \tag{76}$$

where U_0 and $\tilde{U}_0$ are the values of the function U and $\tilde{U}$ when $\vartheta=0$.

The diffraction field of the wave H_{pl} may be represented in the same manner for the wave zone. In the forward half space, i.e., when $\frac{\pi}{2}<\vartheta<\pi$ the Hertz functions of this field are given by the expressions

$$\left.\begin{aligned}
\Pi &= -(-i)^p \frac{2\pi a}{ck^2} B \frac{4\varkappa^2}{\mu_l^2} \frac{\Delta}{1 + \Delta^2} \frac{\sin\frac{\vartheta_l}{2} \sin\frac{\vartheta}{2} J_p(\varkappa \sin\vartheta)}{\sin^2\vartheta} \frac{e^{ikR + \tilde{U}_l + U}}{R} \\
\tilde{\Pi} &= -(-i)^{p+1} \frac{2\pi a}{ck^2} B \frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta_l}{2}} \frac{J_p'(\varkappa \sin\vartheta)}{\sin\vartheta (\cos\vartheta - \cos\vartheta_l)} \frac{1 + \Delta^2 \left(\frac{\operatorname{tg}\frac{\vartheta_l}{2}}{\operatorname{tg}\frac{\vartheta}{2}}\right)^2}{1 + \Delta^2} \frac{e^{ikR + \tilde{U}_l + \tilde{U}}}{R}
\end{aligned}\right\} \tag{77}$$

while in the rear half space we have

$$
\left.\begin{aligned}
\Pi &= -(-i)^{p}\frac{1}{ck^{3}}B\frac{4\varkappa^{2}}{\mu_{l}^{2}}\frac{\Delta}{1+\Delta^{2}}\frac{\sin\frac{\tilde{\vartheta}_{l}}{2}}{\cos\frac{\vartheta}{2}\sin^{2}\vartheta\, H_{p}(\varkappa\sin\vartheta)}\frac{e^{ikR+\tilde{U}_{l}+U}}{R}\\
\hat{\Pi} &= -(-i)^{p+1}\frac{1}{ck^{3}}B\frac{1}{\sin\frac{\tilde{\vartheta}_{l}}{2}\cos\frac{\vartheta}{2}\sin\vartheta\, H_{p}'(\varkappa\sin\vartheta)\left(\cos\vartheta-\cos\tilde{\vartheta}_{l}\right)}\frac{1+\Delta^{2}\left(\dfrac{\operatorname{tg}\frac{\tilde{\vartheta}_{l}}{2}}{\operatorname{tg}\frac{\vartheta}{2}}\right)^{2}}{1+\Delta^{2}}\frac{e^{ikR+\tilde{U}_{l}+\hat{U}}}{R}
\end{aligned}\right\} \quad (78)
$$

where $\tilde{U}_l$ is the value of the function $\tilde{U}$ when $\vartheta=\pi-\tilde{\vartheta}_l$, i.e., when /344

$$\cos\vartheta=\frac{\gamma_l}{\varkappa}.$$

It is interesting to compare the radiation field of wave H_{p_l} in the forward half space given by equation (77), with the field of the same wave (47) given by the Huygens principle. The extract expression for the fields $E_\vartheta=H_\varphi$ differs from (47) by the factor

$$\frac{2\sin\frac{\tilde{\vartheta}_l}{2}\sin\frac{\vartheta}{2}}{1+\cos\tilde{\vartheta}_l\cos\vartheta}e^{U_0-\tilde{U}_0+\tilde{U}_l+U},$$

while for fields $E_\varphi=-H_\vartheta$ it differs by the factor

$$\frac{\sin\frac{\vartheta}{2}}{\sin\frac{\tilde{\vartheta}_l}{2}}\,\frac{1+\Delta^{2}\left(\dfrac{\operatorname{tg}\frac{\tilde{\vartheta}_{l}}{2}}{\operatorname{tg}\frac{\vartheta}{2}}\right)^{2}}{1+\Delta^{2}}e^{\tilde{U}_l+\hat{U}}.$$

For large values of the parameter $\varkappa$ and for angles ϑ and $\tilde{\vartheta}_l$ which are close to π, these factors are close to unity so that the Huygens principle for forward radiation must give good results.

Making the same comparison for the electric wave E_{pl} we can see that the fields $E_\vartheta=H_\varphi$, according to equations (74), differs from these same fields given by the Huygens principle (45) by the factor

$$\frac{\sin\frac{\tilde{\vartheta}_l}{2}}{\sin\frac{\vartheta}{2}}\,\frac{1+\Delta^{2}\left(\dfrac{\operatorname{tg}\frac{\vartheta}{2}}{\operatorname{tg}\frac{\tilde{\vartheta}_{l}}{2}}\right)^{2}}{1+\Delta^{2}}e^{\tilde{U}_l+U},$$

which becomes infinite when $\vartheta=\pi$. The presence of the magnetic Hertz function in the radiation field (i.e., the fields $E_{\varphi}=-H_{\vartheta}$) is not conveyed at all by the Huygens principle; as a matter of fact, under condition (71) the magnetic Hertz function is small (almost for all directions) compared with the electric one, which can be easily seen from equation (77). Generally speaking the radiation of electric waves (which is characterized by small directionality) is not reflected as well by the Huygens principle as the radiation of magnetic fields.

The Huygens principle does not account at all for the radiation field in the rear half space. The physical meaning of expressions for the field in the rear half space under conditions (71) will be considered by us elsewhere.

For the sake of completion we also present equations for the reflection and transformation coefficients in the form analogous to the equations of this section,

$$\left.\begin{aligned}
R_{l,m} &= -i\frac{\sqrt{(\varkappa+\gamma_l)(\varkappa+\gamma_m)}}{2\gamma_m(\gamma_l+\gamma_m)}\left[1-\frac{\Delta^2}{1+\Delta^2}\frac{2\varkappa(\gamma_l+\gamma_m)}{(\varkappa+\gamma_l)(\varkappa+\gamma_m)}\right]e^{U_l+U_m}\\
T_{l,m} &= \frac{i\Delta}{1+\Delta^2}\sqrt{\frac{\varkappa+\check{\gamma}_m}{\varkappa+\gamma_l}}\frac{e^{U_l+\check{U}_m}}{\left(1-\frac{p^2}{\mu_m^2}\right)\check{\gamma}_m}\\
\check{R}_{l,m} &= -\frac{i\mu_m^2}{\sqrt{(\varkappa+\check{\gamma}_l)(\varkappa+\check{\gamma}_m)}\left(1-\frac{p^2}{\mu_m^2}\right)2\check{\gamma}_m(\check{\gamma}_l+\check{\gamma}_m)}\left[1+\frac{\Delta^2}{1+\Delta^2}\frac{2\varkappa(\check{\gamma}_l+\check{\gamma}_m)}{(\varkappa-\check{\gamma}_l)(\varkappa-\check{\gamma}_m)}\right]e^{\check{U}_l+\check{U}_m}\\
\check{T}_{l,m} &= \frac{i\Delta}{1+\Delta^2}\sqrt{\frac{\varkappa+\check{\gamma}_l}{\varkappa+\gamma_m}}\frac{\varkappa^2 e^{\check{U}_l+U_m}}{\mu_l^2\gamma_m}
\end{aligned}\right\}. \tag{79}$$

If we interpret U and $\check{U}$ to mean the integrals (73), then all the equations written out above are exact; computations by means of these equations are rather complicated. However, when conditions (71) are satisfied, the integrals (73) are reduced to the universal function

$$U(s,q)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\left(1-e^{i2\pi q-\frac{t^2}{2}}\right)\frac{dt}{t-se^{i\frac{\pi}{4}}}, \tag{80}$$

which was first introduced in our earlier work (ref. 3).

Specifically

$$U=U(s,q)\qquad \check{U}=U(s,\check{q}), \tag{81}$$

where

$$s=\sqrt{2\varkappa}\cos\vartheta\qquad q=\frac{1}{\pi}\Omega(\varkappa)\qquad \check{q}=\frac{1}{\pi}\check{\Omega}(\varkappa). \tag{82}$$

The substitution (81) usually gives us a sufficiently accurate approximation even for moderate values of $\varkappa$. Thus, when $\varkappa=4$ equations (76) and (81) produce an error for the function Δ (when p=1), which is less than 1 percent.

The derivation of the approximate equations completes the general theory of nonsymmetric waves in a circular wave guide. Numeric results for the more interesting types of waves will be presented elsewhere (ref. 4).

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